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Hamiltonian symplectomorphisms and the Berry phase

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Abstract

On the space $\mathcal{L}$, of loops in the group of Hamiltonian symplectomorphisms of a symplectic quantizable manifold, we define a closed $\mathbf{Z}$-valued 1-form Ω. If Ω vanishes, the prequantization map can be extended to a group representation. On $\mathcal{L}$ one can define an action integral as an $\mathbf{R}/\mathbf{Z}$-valued function, and the cohomology class $[\Omega]$ is the obstruction to the lifting of that action integral to an $\mathbf{R}$-valued function. The form Ω also defines a natural grading on $\pi_1(\mathcal{L})$.

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1. Introduction

In the process of quantization of a symplectic manifold (M, ω) it is necessary to fix a polarization I, then the corresponding quantization $\mathcal{Q}_I$ is the space of the sections of a prequantum bundle L, which are parallel along the leaves of the polarization I [17]. The identification of the $\mathcal{Q}_I$ obtained by fixing different polarizations is one of the goals of the geometric quantization, but "the theory is far from achieving this goal" [3, p. 267]. This issue has been treated in several particular cases: the identification of the quantizations of the moduli space of flat connections on a closed surface has been studied in [1,5]; the case when M is a torus has been treated in [13]. The problems involved in an identification of the spaces $\mathcal{Q}_I$ were analyzed in [14], when the polarizations considered are of type Kähler.

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Here, we consider a similar situation. If $\{\psi_t | t \in [0, 1]\}$ is a Hamiltonian isotopy of M [8] and F a foliation on M, the action of ψ_t produces a family F_t of foliations. We have the spaces $\mathcal{Q}_{F_t}$ of sections of L which are "polarized" with respect to F_t, i.e., sections parallel along the leaves of F_t. We shall construct isomorphisms $\tau \in \mathcal{Q}_F \to \tau_t \in \mathcal{Q}_{F_t}$, which permit us to "transport" the vectors in $\mathcal{Q}_F$ to the spaces $\mathcal{Q}_{F_t}$ in a continuous way. In general this transport has non-vanishing "curvature", i.e., it depends on the isotopy which joins a given symplectomorphism with id.

In the prequantization process of (M, ω) one assigns to each function f on M an operator $\mathcal{P}_f$ [11, pp. 57–59], which acts on the space $\Gamma(L)$ of sections of L. The map $\mathcal{P}$ is a representation of $\mathrm{Vect}_H(M)$, the algebra of Hamiltonian vector fields on M. There are obstructions to extend this representation to a representation of $\mathrm{Ham}(M)$, the group of Hamiltonian symplectomorphisms of M [8]. We analyze the relation between these obstructions and the curvature of the aforementioned transport.

If the Hamiltonian isotopy ψ_t is a loop in the group $\mathrm{Ham}(M)$ and N is a Lagrangian leaf of the foliation F, then $\psi_t(N)$ is a loop of submanifolds of M and the corresponding Berry phase is defined [16]. We prove the existence of a number $\kappa(\psi) \in U(1)$, which depends only on the loop ψ, and that relates *any section* ρ with ρ_1, the section resulting of the transport of ρ, by the formula $\rho_1 = \kappa(\psi)\rho$. So $\kappa(\psi)$ is the "holonomy" of the transport along ψ. It turns out that the holonomy of our transport is essentially the Berry phase of the loop $\psi_t(N)$. Using the map κ we construct on $\mathcal{L}$, the space of loops in $\mathrm{Ham}(M)$ based at id, a closed 1-form Ω. The vanishing of Ω is equivalent to the invariance of the Berry phase under deformations of the loop ψ. We will prove that there is a well-defined an $\mathbb{R}/\mathbb{Z}$-valued action integral on $\mathcal{L}$. The exactness of Ω is equivalent to the existence of a lift of the action integral to an $\mathbb{R}$-valued map. The integral of the form Ω along a loop ϕ^s in $\mathcal{L}$ is in fact the winding number of the map $s \in S^1 \mapsto \kappa(\phi^s) \in U(1)$, so Ω is $\mathbb{Z}$-valued. This property permits to define a grading on $\pi_2(\mathrm{Ham}(M))$ compatible with the group structure.

In Section 2 is introduced the transport of vectors $\tau \in \mathcal{Q}_F$ to vectors $\tau_t \in \mathcal{Q}_{F_t}$. Such a transport is determined by the differential equation which it generates, i.e.,

$$\frac{\mathrm{d}\tau_t}{\mathrm{d}t} = \zeta(F_t, \tau_t),$$

where ζ is a section of L. The condition $\tau_t \in \mathcal{Q}_{F_t}$ gives rise to an equation for ζ. This equation does not determine uniquely ζ, however it is possible to choose a *natural* solution for ζ using the time-dependent Hamiltonian f_t which generates the isotopy. If the isotopy is closed, i.e., $\psi_1 = \mathrm{id}$, given a leaf N of F, it is easy to show the existence of a constant κ such that $\tau_{1|N} = \kappa\tau_{|N}$ for all $\tau \in \mathcal{Q}_F$. So one can define the holonomy for the transport of such sections $\tau_{|N}$. In this Section we also study the relation between this holonomy and the Berry phase of the loop $\psi_t(N)$ of Lagrangian submanifolds of M.

Section 3 is concerned with the properties of $\kappa(\psi)$. First we prove its existence and determine its expression in terms of the Hamiltonian function and the symplectic form. Given $\{\psi_t | t \in [0, 1]\}$ a loop in $\mathrm{Ham}(M)$, if q is a point of M, then the *general* action integral around the closed curve $\psi_t(q)$ is $\int_S \omega$, where S is any 2-submanifold bounded by the curve $\psi_t(q)$. However, for these particular curves one can also define the action integral

$$\mathcal{A}(\psi(q)) = \int_S \omega - \int_0^1 f_t(\psi_t(q))\,\mathrm{d}t. \tag{1.1}$$

$\mathcal{A}(\psi(q))$ is well-defined considered as an element of $\mathbb{R}/\mathbb{Z}$. Using results of Section 2 about the transport of polarized sections we will prove that $\mathcal{A}(\psi(q))$ is *independent* of the point $q \in M$ and that $\kappa(\psi) = \exp(2\pi \mathrm{i}\mathcal{A}(\psi))$.

In [15] Weinstein defined a representation $\mathbf{A}$ of $\pi_1(\mathrm{Sym}(M))$ as follows. $\mathbf{A}(\psi)$ is the mean value over q of the general action integrals around the curves $\psi_t(q)$. When ψ_t is a 1-parameter subgroup generated by a Hamiltonian function f, $\mathbf{A}$ and κ are related by $\kappa(\psi) = \exp(2\pi \mathrm{i}\mathbf{A}(\psi))$, assumed that the Hamiltonian function f is normalized so that $\int f\omega^n = 0$. The domain of the map κ is less general than the domain of $\mathbf{A}$, however the restriction to the Hamiltonian symplectomorphisms allows us to introduce the second summand in (1.1), so we obtain an invariant without averaging on M, i.e., in contrast with $\mathbf{A}(\psi)$ the value $\kappa(\psi)$ can be calculated pointwise. κ is not invariant under homotopies; this fact has an interesting meaning. One can define a 1-form on $\mathcal{L}$ as follows. Given a curve ψ^s in $\mathcal{L}$, and denoting by Z the vector field defined by this curve, the action of the 1-form Ω on Z is given by

$$\Omega(Z) = -\frac{1}{2\pi \mathrm{i}}\frac{\mathrm{d}}{\mathrm{d}s}(\log(\kappa(\psi^s))).$$

Hence, the vanishing of Ω is equivalent to the invariance of $\kappa(\psi)$ with respect to deformations of the isotopy ψ. The property $\Omega = 0$ is also a sufficient condition for $\mathcal{P}$ extends to a representation of $\widetilde{\mathrm{Ham}}(M)$, the universal cover of $\mathrm{Ham}(M)$.

In Section 4, we prove that Ω is a closed 1-form that defines an element of $H^1(\mathcal{L}, \mathbb{Z})$. We will also find a simple interpretation of the cohomology class of Ω; it is the obstruction for the lifting of $\mathcal{A}$ to an $\mathbb{R}$-valued map. The identification of $\pi_1(\mathcal{L})$ with $\pi_2(\mathrm{Ham}(M))$ will allow us to define a grading on the group $\pi_2(\mathrm{Ham}(M))$ by means of the form Ω.

In Section 5, we consider as symplectic manifold a coadjoint orbit of the group $SU(2)$. There are orbits $\mathcal{O}$ diffeomorphic to S^2 and for these manifolds it is easy to determine the value of κ on the loops which are 1-parameter subgroups in $\mathrm{Ham}(\mathcal{O})$. With this example we check the general properties of κ stated in Section 3.

2. Loops of submanifolds and the Berry phase

Let M be a connected, compact, symplectic C^∞ manifold of dimension $2n$, with symplectic form ω. Let us suppose that (M, ω) is *quantizable*, in other words, we assume that ω defines a cohomology class in $H^2(M, \mathbb{R})$ which belongs to the image of $H^2(M, \mathbb{Z})$ in $H^2(M, \mathbb{R})$ [17, p. 158]. Then there exists a smooth Hermitian line bundle on M whose first Chern class is $[\omega]$, and on this bundle is defined a connection D compatible with the Hermitian structure and whose curvature is $-2\pi \mathrm{i}\omega$. The bundle and the connection are not uniquely determined by ω. The family of all possible pairs (line bundle, connection) can be labelled by the elements of $H^1(M, U(1))$ [17, p. 161]. From now on we suppose that a "prequantum bundle" L and a connection D have been fixed, unless it is otherwise indicated.

Let F be a foliation on M. If τ is a C^∞ section of L such that $D_A\tau = 0$, for all $A \in F$, then τ is called an *F-polarized* section, and the space of F-polarized sections of L is denoted by $\mathcal{Q}_F$.

Let $\{\psi_t | t \in [0,1]\}$ be the Hamiltonian isotopy in M generated by the time-dependent Hamiltonian function f_t, i.e.,

$$\frac{\mathrm{d}\psi_t}{\mathrm{d}t} = X_t \circ \psi_t, \qquad \iota_{X_t}\omega = -\mathrm{d}f_t, \qquad \psi_0 = \mathrm{id}.$$

Then, for each, t we have a distribution $F_t := (\psi_t)_*(F)$. Moreover, if N is an integral submanifold of F then $N_t := \psi_t(N)$ is an integral submanifold of F_t. The family N_t is an isodrastic deformation of N [16].

Given τ an F-polarized section of L, we want to define a continuous family τ_t of sections of L such that $\tau_0 = \tau$ and τ_t is F_t-polarized for all t. The continuity condition means that there is a section ζ of L such that

$$\tau_{t+s} = \tau_t + s\zeta(\tau_t) + \mathrm{O}(s^2), \tag{2.1}$$

where $\mathrm{O}(s^2)$ is relative to the uniform C^1-norm in the space $\Gamma(M, L)$ of C^∞ sections of L. We will see the restrictions on ζ involved by the continuity condition (2.1), but first of all we start with a previous result.

Given the isotopy ψ_t, each section ρ of L determines a family ρ_t of sections by the equation

$$\frac{d\rho_t}{dt} = -D_{X_t}\rho_t - 2\pi \mathrm{i} f_t\rho_t, \quad \rho_0 = \rho. \tag{2.2}$$

Proposition 1. *Let A be a vector field on M. If the family ρ_t of sections of L satisfies* (2.2), *then $D_A\rho = 0$ implies $D_{A_t}\rho_t = 0$ for $A_t = (\psi_t)_*(A)$.*

Proof. For a fixed t one has

$$\left(\frac{\mathrm{d}\psi_{t'}(q)}{\mathrm{d}t'}\right)\bigg|_{t'=t} = X_t(\psi_t(q)).$$

If we put $t' = t + s$ and $\phi_s := \psi_{t+s} \circ \psi_t^{-1}$, then

$$\left(\frac{\mathrm{d}\phi_s(p)}{\mathrm{d}s}\right)\bigg|_{s=0} = X_t(p). \tag{2.3}$$

As $\{\phi_s\}$ satisfies (2.3), then for the vector field $A'_s = (\phi_s)_*(A_t)$, we have

$$A'_s = A_t - s[X_t, A_t] + \mathrm{O}(s^2). \tag{2.4}$$

Since $A_{t+s} = A'_s$ one has

$$\dot{A}_t := \left(\frac{\mathrm{d}A_{t'}}{\mathrm{d}t'}\right)\bigg|_{t'=t} = \left(\frac{\mathrm{d}A'_s}{\mathrm{d}s}\right)\bigg|_{s=0} = -[X_t, A_t]. \tag{2.5}$$

On the other hand

$$\frac{\mathrm{d}}{\mathrm{d}t}(D_{A_t}\rho_t) = D_{\dot{A}_t}\rho_t + D_{A_t}(-D_{X_t}\rho_t - 2\pi\mathrm{i}f_t\rho_t). \tag{2.6}$$

As the curvature of D is $-2\pi\mathrm{i}\omega$

$$-D_{A_t}D_{X_t}\rho_t - D_{[X_t,A_t]}\rho_t = -2\pi\mathrm{i}\omega(X_t, A_t)\rho_t - D_{X_t}D_{A_t}\rho_t. \tag{2.7}$$

Since $\iota_{X_t}\omega = -\mathrm{d}f_t$, by (2.5) from (2.6) and (2.7) it follows

$$\frac{\mathrm{d}}{\mathrm{d}t}(D_{A_t}\rho_t) = -D_{X_t}(D_{A_t}\rho_t) - 2\pi\mathrm{i}f_t D_{A_t}\rho_t. \tag{2.8}$$

This is a first-order differential equation for the section $\xi(t) := D_{A_t}\rho_t$; if $\xi(0) = D_A\rho$ is zero, then $D_{A_t}\rho_t = 0$ for every t by the uniqueness of solutions. □

Given $\rho \in \Gamma(L)$, the family $\{\rho_t\}$ which satisfies Eq. (2.2) defines a "transport" of ρ, "along the isotopy" $\psi = \{\psi_t\}$, with the property that $\rho_t \in \mathcal{Q}_{F_t}$ if $\rho \in \mathcal{Q}_F$. The time-1 section ρ_1 will be denoted by $\mathcal{T}_\psi(\rho)$.

By Proposition 1 as section ζ in (2.1) can be taken

$$\zeta(\tau_t) = -D_{X_t}\tau_t - 2\pi\mathrm{i}f_t\tau_t. \tag{2.9}$$

In general this is not the unique possibility for ζ. In fact if A is a vector field with $A_p \in F(p) \subset T_pM$, using (2.4) and (2.1) one has

$$D_{A'_s}\tau_{t+s} = D_{A_t}\tau_t + s(D_{A_t}\zeta - D_{[X_t,A_t]}\tau_t) + \mathrm{O}(s^2).$$

As $A'_s = (\psi_{t+s})_*(A) \in F_{t+s}$, the conditions $D_{A'_s}\tau_{s+t} = 0$, and $D_{A_t}\tau_t = 0$ imply

$$D_{A_t}\zeta = D_{[X_t,A_t]}\tau_t \quad \text{for every } \ A_t \in F_t. \tag{2.10}$$

This is the equation for ζ, and it is straightforward to check that the ζ defined in (2.9) satisfies (2.10).

The solution (2.9) and (2.10) will be called the "natural" solution and the transport defined by (2.2) the "natural" transport.

Let $\{\psi_t | t \in [0, 1]\}$ and $\{\tilde{\psi}_t | t \in [0, 1]\}$ be two isotopies with $\psi_1 = \tilde{\psi}_1$. We have $\mathcal{T}_\psi(\rho)$ and $\mathcal{T}_{\tilde{\psi}}(\rho)$, the sections resulting of the transport of ρ along both these isotopies. In general $\mathcal{T}_\psi(\rho)$ and $\mathcal{T}_{\tilde{\psi}}(\rho)$ will not be equal, i.e., the natural transport is not flat. In Section 3, we will analyze the corresponding "curvature".

The operator $-D_{X_t} - 2\pi\mathrm{i}f_t$ can be considered from another point of view. One can associate to each C^∞ function f on M a linear operator $\mathcal{P}_f$ on the space $\Gamma(L)$, defined by

$$\mathcal{P}_f(\sigma) = -D_{X_f}\sigma - 2\pi\mathrm{i}f\sigma,$$

where X_f is the Hamiltonian vector field determined by f. It is easy to check $\mathcal{P}_{\{f,g\}} = \mathcal{P}_f \circ \mathcal{P}_g - \mathcal{P}_g \circ \mathcal{P}_f =: [\mathcal{P}_f, \mathcal{P}_g]$, where the Poisson bracket $\{f, g\}$ is defined as $\omega(X_g, X_f)$. So $\mathcal{P}$ is a representation of the Lie algebra $C^\infty(M)$, the *prequantization representation* [11]. On the other hand, in the algebra of linear operators on $\Gamma(L)$ one can consider the ideal $\mathbb{C}$ consisting of the operators multiplication by a constant, this allows us to define a representation of the algebra Lie(Ham(M)) in the algebra End$(\Gamma(L))/\mathbb{C}$. It is reasonable to conjecture the existence of obstructions to extend the above representation to a projective representation

$$\mathrm{Ham}(M) \to PL(\Gamma(L))$$

of the group Ham(M). In Section 3, we will relate these obstructions with the curvature of the natural transport.

2.1. Relation with the Berry phase

The connection on the $\mathbb{C}^{\times}$-principal bundle $L^{\times} = L - \{\text{zerosection}\}$, associated to the prequantum bundle L, will be denoted by α. Given $c \in \mathbb{C}$, the vertical vector field on $L^{\times}$ generated by c will be denoted by W_c, i.e., $W_c(q)$ is the vector defined by the curve in $L^{\times}$ given by $q \cdot \mathrm{e}^{2\pi \mathrm{i} ct}$.

Henceforth in this section, we assume that F is a Lagrangian foliation. Given $\tau \in \mathcal{Q}_F$, as τ is parallel along the leaves of the distribution F, if N is a leaf of F and if $\tau_{|N} \neq 0$, then $\tau(p) \neq 0$ for all $p \in N$. So $\tau(N)$ is a Planckian submanifold [12] of $L^{\times}$ over N.

The proof of the following Lemma is straightforward.

Lemma 2. *If $X \in T_m N$ and $\tau \in \mathcal{Q}_F$, the vector $\tau_*(X) \in T_q L^{\times}$, where $q = \tau(m)$, satisfies $\tau_*(X) = H(X)(q) + (D_X \tau)(m)$, with $H(X)(q)$ the horizontal lift of X at the point q.*

Given a Hamiltonian isotopy ψ_t and $\tau \in \mathcal{Q}_F$, let τ_t be the family generated by the transport of τ along ψ_t. If $p \in N$ one can consider in $L^{\times}$ the following curve:

$$t \to \tau_t(\psi_t(p)).$$

Proposition 3. *The tangent vector defined by $\{\tau_t(\psi_t(p))\}_t$ at $q = \tau_u(\psi_u(p))$ is*

$$H(X_u)(q) + W_{-f_u(\pi(q))}(q).$$

Proof. For t in a small neighborhood of u, as

$$\frac{\mathrm{d}\tau_t}{\mathrm{d}t} = -2\pi \mathrm{i} f_t \tau_t - D_{X_t} \tau_t,$$

one has

$$\begin{aligned}\tau_t(\psi_t(p)) = {} & \tau_u(\psi_t(p)) - (t-u)(2\pi \mathrm{i} f_u(\psi_t(p))\tau_u(\psi_t(p)) \\ & + (D_{X_u}\tau_u)(\psi_t(p))) + \mathrm{O}((t-u)^2).\end{aligned}$$

This curve defines at $t = u$ the following vector of $T_q L^{\times}$

$$(\tau_u)_*(X_u(s)) - (2\pi \mathrm{i} f_u(s)\tau_u(s) + (D_{X_u}\tau_u)(s)), \qquad (2.11)$$

where $s := \psi_u(p)$. As $\tau_u(s) = q$ by Lemma 2 $(\tau_u)_*(X_u(s)) = H(X_u(s))(q) + (D_{X_u}\tau_u)(s)$. So the expression (2.11) is equal to

$$H(X_u(\pi(q)))(q) - W_{f_u(\pi(q))}(q).$$

In short, the tangent vector at q defined by the curve considered is $H(X_u) + W_{-f_u}$. □

We will use the following simple Lemma.

Lemma 4. *If N is a connected submanifold of M and σ and ρ are sections of L parallel along N, where ρ non-identically zero on N, then $\sigma_{|N} = k\rho_{|N}$, with k constant.*

Proof. As ρ is parallel along N, $\rho(x) \neq 0$ for all $x \in N$, so there is a function h on N with $\sigma_{|N} = h\rho_{|N}$. The relation

$$D_A(\sigma_{|N}) = A(h)\rho_{|N} + hD_A(\rho_{|N})$$

for every $A \in TN$, implies that h is constant on N. □

Given $\tau \in \mathcal{Q}_F$, and ψ_t a Hamiltonian *closed* isotopy, i.e., such that $\psi_1 = \mathrm{id}$, then $\mathcal{T}_\psi(\tau)$ is also F-polarized. If N is a leaf of F and $\tau_{|N} \neq 0$ by Lemma 4

$$\mathcal{T}_\psi(\tau)_{|N} = \kappa\tau_{|N}, \tag{2.12}$$

where κ is a constant. From linearity of the transport and Lemma 4 it follows that κ is independent of the section τ. Hence κ can be considered as the *holonomy* of the natural transport, along the closed isotopy ψ_t, of F-polarized sections of $L_{|N}$. In Section 3, we will prove the existence of holonomy for the transport of *arbitrary* sections of L.

Now we recall some results of Weinstein about the Berry phase (for details see [16, p. 142]). If $\{N_t\}_t$ is a loop of Lagrangian submanifolds generated by the closed isotopy ψ_t. Let ϵ_t be a smooth density on N_t such that $\int_{N_t} f_t\epsilon_t = 0$. Let $\{r_t\}$ be the family of isomorphisms of $(L^\times_{|N}, \alpha)$ to $(L^\times_{|N_t}, \alpha)$ determined by $\{f_t\}$, i.e., the isomorphisms generated by the vector fields

$$H(X_t) + W_{-f_t}, \tag{2.13}$$

where $H(X_t)$ is the horizontal lift of X_t. The submanifold $r_1(\tau(N))$ "differs" from $\tau(N)$ by and element $\theta \in U(1)$, i.e.,

$$r_1(\tau(N)) = \theta\tau(N). \tag{2.14}$$

If we denote by hol the holonomy on N defined by de-connection α, $\mathrm{hol} : \pi_1(N) \to U(1)$, then the Berry phase of the family (N_t, ϵ_t) of weighted submanifolds is the class of θ in the quotient $U(1)/(\mathrm{Im(hol)})$. Up to here the results of Weinstein.

Theorem 5. *If ψ_t is a closed Hamiltonian isotopy and N a connected leaf of the Lagrangian foliation F, then the Berry phase of (N_t, ϵ_t), with $N_t = \psi_t(N)$, is the class in $U(1)/(\mathrm{Im(hol)})$ of the holonomy of the natural transport along ψ_t* of *F-polarized sections of $L_{|N}$.*

Proof. Given $p \in N$, by Proposition 3 and (2.13) the curves in $L^\times\{\tau_t(\psi_t(p)\}_t$ and $\{r_t(\tau(p))\}_t$ define the same vector field. As they take the same value for $t = 0$, it turns out that $r_t(\tau(p)) = \tau_t(\psi_t(p))$ for all $p \in N$, hence the above complex θ in (2.14) is determined by

$$\tau_1(p) = \theta\tau(p). \tag{2.15}$$

From (2.12) and (2.15) we conclude $\theta = \kappa$. □

As we said the Berry phase of a loop of Lagrangian submanifolds was introduced in [16], now we shall relate the holonomy of the natural transport with Berry's phase such as it is defined in quantum mechanics [10].

Let f_t be a time-dependent Hamiltonian for a physical system with $f_0 = f_1$. We denote by $\hat{f}_t$ the corresponding quantum operators. We assume that the lowest eigenvalue $E(t)$ for $\hat{f}_t$ is non-degenerate, and it is isolated from the rest of the spectrum of $\hat{f}_t$. For simplicity we suppose that $E(t) = 0$. Let φ_0 be a unit eigenvector for $E(0)$. The state φ_0 will change with t according to time-dependent Schrödinger equation $\mathrm{i}(\partial\varphi_t/\partial t) = \hat{f}_t(\varphi_t)$. The adiabatic theorem asserts that φ_t will be also an eigenvector of the lowest energy of $\hat{f}_t$, assumed that the adiabatic approximation is valid [2,9]. Since $f_0 = f_1$, φ_1 will be equal to φ_0 except for a phase factor, $\varphi_1 = \mathrm{e}^{\mathrm{i}\gamma}\varphi_0$, and $\mathrm{e}^{\mathrm{i}\gamma}$ is the Berry phase [10].

In the frame of geometric quantization, the space $\mathcal{Q}$ of states is the set of sections of the prequantum bundle L which are covariantly constant along the leaves of a fixed Lagrangian polarization I. Let $\{\psi_t\}_{t\in[0,1]}$ be the closed isotopy generated by f_t, and we denote by $\mathcal{Q}_t$ the space of sections polarized with respect to the distribution $\psi_t(I)$. The family $\{\psi_t\}$ determines a set $\{r_t\}$ of automorphisms of the bundle $(L^\times, \alpha)$, where r_t covers ψ_t. The transformations r_t are determined by the vector fields $H(X_t) + W_{-f_t}$ (see also [11, Chapter 3]). If $\tau^\sharp : L^\times \to \mathbb{C}$ denotes the equivariant map determined by the section τ of L, the action of r_t can be expressed as $(r_t(\tau))^\sharp = \tau^\sharp \circ r_t^{-1}$, i.e., r_t acts by pulling back by r_t^{-1}. On the other hand, the temporal evolution of the state $\tau \in \mathcal{Q}$ is $\tau_t = r_t(\tau) \in \mathcal{Q}_t$ (see [17, p. 201]). If the adiabatic approximation holds, τ_t will be a unit lowest eigenvector of $\hat{f}_t$, assumed that τ is a unit lowest eigenvector of $\hat{f}_0$. As $\hat{f}_0 = \hat{f}_1$, then $\tau_{1|N} = \mathrm{e}^{\mathrm{i}\gamma}\tau_{|N}$, where N is any leaf of the polarization $I = I_1$. It follows from (2.12) that $\mathrm{e}^{\mathrm{i}\gamma}$ is just the holonomy of the natural transport.

3. The holonomy of the natural transport

We will prove that it makes sense to define the holonomy of the natural transport of *arbitrary* sections of L. We start with $\psi = \{\psi_t | t \in [0, 1]\}$ a *closed* Hamiltonian isotopy, generated by the time-dependent Hamiltonian function f_t, i.e., ψ is a loop at id in the group $\mathrm{Ham}(M)$. So we must consider the corresponding Eq. (2.2) and study its solution. Let μ be a local frame for L, defined on $R \subset M$, and β the connection form in this frame. There is a time-dependent function $g(t, .)$ such that

$$\rho_t(p) = g(t, p)\mu(p), \quad t \in [0, 1], \quad p \in R. \tag{3.1}$$

Hence (2.2) can be written

$$\frac{\partial}{\partial t} g(t, .) = -X_t(g(t, .)) - \beta(X_t) g(t, .) - 2\pi \mathrm{i} f_t g(t, .),$$
$$g(0, p) = p \quad \text{for all} \quad p \in R. \tag{3.2}$$

Fix a point $q \in M$, we put $\sigma(t) := \psi_t(q)$. Assumed that the closed curve σ is contained in R, Eq. (3.2) on the points of this curve is

$$\frac{\partial g}{\partial t}(t,\sigma(t)) + X_t(\sigma(t))(g(t,.))$$
$$= -\beta_{\sigma(t)}(X_t)g(t,\sigma(t)) - 2\pi\mathrm{i} f_t(\sigma(t))g(t,\sigma(t)). \tag{3.3}$$

The second summand on the left-hand side is the action of the vector $X_t(\sigma(t))$ on the function $g(t,.) : M \to \mathbb{R}$. If we consider the curve $\hat{\sigma} : [0,1] \to \mathbb{R} \times M$, defined by $\hat{\sigma}(t) = (t, \sigma(t))$ and we put $\hat{g}(t) := g(\hat{\sigma}(t))$, Eq. (3.3) can be written

$$\frac{\mathrm{d}\hat{g}}{\mathrm{d}t} = -\beta_{\sigma(t)}(X_t)\hat{g}(t) - 2\pi\mathrm{i} f_t(\sigma(t))\hat{g}(t). \tag{3.4}$$

Hence

$$\hat{g}(t) = g(0,q)\exp\left(\int_0^t (-\beta_{\sigma(t')}(X_{t'}) - 2\pi\mathrm{i} f_{t'}(\sigma(t')))\,\mathrm{d}t'\right).$$

As the closed curve σ is nullhomologous [8, p. 334], let S be any oriented 2-submanifold bounded by the closed curve σ, then

$$\int_0^1 \beta_{\sigma(t)}(X_t)\,\mathrm{d}t = \int_S \mathrm{d}\beta.$$

As the curvature of L is $-2\pi\mathrm{i}\omega$, we have

$$\hat{g}(1) = g(0,q)\exp\left(2\pi\mathrm{i}\int_S \omega - 2\pi\mathrm{i}\int_0^1 f_t(\psi_t(q))\,\mathrm{d}t\right). \tag{3.5}$$

Given the loop ψ in $\mathrm{Ham}(M)$, the Hamiltonian vector fields X_t determine the Hamiltonian f_t up to an additive constant. In certain cases it is possible to fix this Hamiltonian function in a natural way, for instance when X_t is an invariant vector field on a coadjoint orbit. In a general case f_t can be fixed by imposing the condition that f_t has zero mean with respect to the canonical measure on M induced by ω; henceforth we assume that f_t satisfies this *normalization condition*.

For p point in M one defines

$$\kappa_p(\psi) := \exp\left(2\pi\mathrm{i}\int_S \omega - 2\pi\mathrm{i}\int_0^1 f_t(\psi_t(p))\,\mathrm{d}t\right), \tag{3.6}$$

where S is any surface bounded by the closed curve $\psi_t(p)$ in M.

Given the closed isotopy ψ, one can define the *action integral* [8,15] $\mathcal{A}(\psi)(p)$ around the curve $\psi_t(p)$ as the element of $\mathbb{R}/\mathbb{Z}$ determined by $(2\pi\mathrm{i})^{-1}$ times the exponent of (3.6). Hence $\kappa_p(\psi) = \exp(2\pi\mathrm{i}\mathcal{A}(\psi)(p))$.

If the Hamiltonian function is independent of t (i.e., the loop ψ is 1-parameter subgroup in $\mathrm{Ham}(M)$), then it is constant along $\psi_t(p)$. Consequently the second integral in (3.6) is equal to $f(p)$.

From (3.1) it follows $\rho_1(q) = \kappa_q(\psi)\rho(q)$. And by choosing appropriate local frames, one can prove for any $\rho \in \Gamma(L)$

$$\rho_1(p) = \kappa_p(\psi)\rho(p) \quad \text{for any} \quad p \in M. \tag{3.7}$$

To study the function $\kappa(\psi) : M \to U(1)$ we will use properties of the sections of L polarized with respect to certain foliations. There may be topological obstructions to the existence of such foliations, however we will prove some properties of that function using the existence of families of vector fields which define foliations in parts of M.

Let $\mathcal{B} := \{B_1, \dots, B_m\}$ be a set of vector fields on M which define an m-dimensional foliation on $M - K$, where K is a subset of M. This foliation will be denoted also by $\mathcal{B}$. We put

$$\mathcal{B}_t = \{B_j(t) := (\psi_t)_*(B_j)\}_{j=1,\dots,m},$$

and this set defines a foliation on $M - \psi_t(K)$. Moreover if N is a leaf of $\mathcal{B}$, then $N_t = \psi_t(N)$ is a leaf of $\mathcal{B}_t$. On the other hand, according to Proposition 1, if τ is a section of L which is $\mathcal{B}$-polarized, i.e., such that $D_{B_j}\tau = 0$, $j = 1, \dots, m$, then the section τ_t solution to (2.2), is $\mathcal{B}_t$-polarized.

Let $N \subset M - K$ be a *connected* integral submanifold of $\mathcal{B}$. Given τ a $\mathcal{B}$-polarized section of $L_{|N}$ with τ non-identically zero on N, then $\tau_1(p) = \kappa_p(\psi)\tau(p)$ for $p \in N$. As τ_1 and τ are $\mathcal{B}$-polarized by Lemma 4 one deduces that $\kappa_p(\psi)$ is independent of the point $p \in N$. The above results can be summarized in the following.

Proposition 6. *Let (M, ω) be a compact, quantizable manifold, and ψ a loop in* $\mathrm{Ham}(M)$ *at* id. *If p, q are points which belong to a connected integral submanifold N of the foliation on $M - K$ defined by $\mathcal{B}$, then $\kappa_q(\psi) = \kappa_p(\psi)$, provided that there is a $\mathcal{B}$-polarized section of L non-zero on N.*

Corollary 7. *If N^i $(i = 1, 2)$ is a connected integral submanifold of $\mathcal{B}^i$, and τ^i a $\mathcal{B}^i$-polarized section of $L_{|N^i}$ with $\tau^i \neq 0$. Then $\kappa_{q_1} = \kappa_{q_2}$, if $q_i \in N^i$ and $N^1 \cap N^2 \neq \emptyset$.*

Proof. If $p \in N^1 \cap N^2$, then for any loop ψ one has $\kappa_{q_1}(\psi) = \kappa_p(\psi) = \kappa_{q_2}(\psi)$. □

On the other hand, if N is a simply connected integral submanifold of an isotropic foliation $\mathcal{B}$, as $\omega_{|TN} = 0$, the parallel transport determined by the connection of L allows us to define a non-zero section ρ of $L_{|N}$ parallel along N. This fact permits other formulation of Proposition 6 without assuming the existence of the non-zero polarized section.

Proposition 8. *Let us suppose that (M, ω) is a compact, quantizable manifold, and that p and q are two points which belong to a connected integral submanifold N of the isotropic foliation $\mathcal{B}$, if N can be written as a finite union of open simply connected subsets, then $\kappa_p = \kappa_q$.*

Proof. It is a consequence of the preceding remark and Corollary 7. □

Corollary 7 admits also a similar version without supposing the existence of τ^i, if we assume that N^i can be expressed as a finite union of simply connected open subsets. So, we have the following proposition.

Proposition 9. *Assumed that (M, ω) is a compact and quantizable manifold. Let $\mathcal{B}^i$ $(i = 1, 2)$ be two sets of vector fields which define isotropic foliations on $M - K^i$, and N^i*

a connected integral submanifold of $\mathcal{B}^i$, *if* N^i *can be written as a finite union of simply connected open subsets and* $N^1 \cap N^2 \neq \emptyset$, *then* $\kappa_{q_1} = \kappa_{q_2}$, *for* $q_i \in N^i$.

Let Y be a transversal vector field on M, i.e., Y is a section of TM which is transversal to the zero section of TM. Then, the Euler class $e(M) \in H^{2n}(M)$ of M is Poincaré dual of the zero locus of Y; so this zero locus is a finite set K of points of M. And from the transversality theory we conclude that this property is also valid for any "generic" vector field. Each point of $M - K$ belongs to a non-constant integral curve of Y. If p and q are two arbitrary points in M one can choose generic vector fields $Y_1, \dots, Y_m$ on M such that p and q can be joined by a path which is the juxtaposition of curves, each of which is a non-constant integral curve of some Y_j. As the curves are isotropic submanifolds of M by Proposition 9, $\kappa_p = \kappa_q$.

If ξ and ψ are two loops in $\mathrm{Ham}(M)$ based at id we can define $\xi \cdot \psi$ as the loop given by the usual product of paths, it is immediate to check that $\kappa(\xi \cdot \psi) = \kappa(\xi)\kappa(\psi)$.

By (3.7) and the foregoing reasoning, we can state the following theorem.

Theorem 10. *If* (M, ω) *is compact and quantizable, the correspondence*

$$\kappa : \{\textit{Loops in}\, \mathrm{Ham}(M)\, \textit{based at}\, \mathrm{id}\} \to U(1)$$

defined by

$$\kappa(\psi) = \exp\left(2\pi \mathrm{i} \int_S \omega - 2\pi \mathrm{i} \int_0^1 f_t(\psi_t(q))\, \mathrm{d}t\right),$$

q being an arbitrary point of M and S any surface bounded by the closed curve $\{\psi_t(q)\}$, *is a well-defined map which satisfies* $\kappa(\xi \cdot \psi) = \kappa(\xi)\kappa(\psi)$. *Moreover*

$$\mathcal{T}_\psi \rho = \kappa(\psi)\rho \tag{3.8}$$

for any section ρ *of the prequantum bundle* L.

By (3.8) it makes sense to call $\kappa(\psi)$ the *holonomy* of the natural transport along the loop ψ.

Corollary 11. *The action integral* $\mathcal{A}(\psi)(p)$ *is independent of* p.

Corollary 12. *Let* f *be a Hamiltonian function such that it defines a* 1*-parameter loop* $\{\psi_t | t \in [0, 1]\}$ *of symplectomorphisms; if p is critical point of* f, *then* $\kappa(\psi) = \exp(-2\pi \mathrm{i} f(p))$. *If p and q are critical points of* f *then* $f(p) = f(q) (\mathrm{mod}\, \mathbb{Z})$.

Proof. As $\psi_t(p) = p$ for all t, the corollary is a consequence of (3.6). □

This relation among the critical values of f has been proved in [15] using the invariant **A** mentioned in Section 1.

The relation (3.8) and Theorem 5 imply the following corollary.

Corollary 13. *Let* $\{N_t := \psi_t(N)\}$, *with* N *a connected, simply connected Lagrangian submanifold of* M *and* $\psi = \{\psi_t\}$ *a loop in* $\mathrm{Ham}(M)$, *then the Berry phase of the family* (N_t, ϵ_t) *of weighted submanifolds is* $\kappa(\psi)$.

Next we will study the behavior of $\kappa(\psi)$ under C^1-deformations of ψ. Let $\psi = \{\psi_t | t \in [0,1]\}$ be a loop in $\mathrm{Ham}(M)$ with $\psi_0 = \psi_1 = \mathrm{id}$. We consider the derivative of $\kappa(\psi^s)$ with respect to the parameter s in a deformation ψ^s of ψ, i.e., $\psi^s = \{\psi_t^s | t \in [0,1]\}$ is an isotopy with $\psi_0^s = \psi_1^s = \mathrm{id}$ generated by the time-dependent Hamiltonian f_t^s; furthermore we assume that $\psi^0 = \psi$. By $\{X_t^s\}_t$ is denoted the family of Hamiltonian vector fields defined by $\{f_t^s\}_t$.

For $q \in M$ we put $\sigma^s(t) := \psi_t^s(q)$, so $\{\sigma^s(t) | t \in [0,1]\}$ is a closed curve and then

$$\kappa(\psi^s) = \exp\left(2\pi \mathrm{i} \int_{S^s} \omega - 2\pi \mathrm{i} \int_0^1 f_t^s(\sigma^s(t))\right) =: \exp(2\pi \mathrm{i} \Delta(s)),$$

where S^s is a surface bounded by the curve σ^s. We set

$$X_t := X_t^0, \qquad f_t := f_t^0, \qquad \sigma(t) = \sigma^0(t).$$

The variation of $\sigma^s(t)$ with s permits to define the vector fields Y_t, i.e.,

$$Y_t(\sigma^s(t)) := \frac{\partial}{\partial s}\sigma^s(t). \tag{3.9}$$

For an "infinitesimal" s the curves σ^l, with $l \in [0,s]$ determine the "lateral surface" J of one "wedge" (see Fig. 1) whose base and cover are the surfaces S and S^s, respectively. The ordered pairs of vectors $(X_t(\sigma(t)), Y_t(\sigma(t)))$ fix an orientation on J, which in turn determines an orientation on the closed surface $T = S \cup J \cup S^s$. If we assume that S and S^s are oriented by means of the orientations of curves σ and σ^s, from the fixed orientation on T it follows $T = J - S + S^s$.

As ω satisfies the integrality condition

$$-\int_S \omega + \int_{S^s} \omega = -\int_J \omega \quad (\mathrm{modulo}\, \mathbb{Z}). \tag{3.10}$$

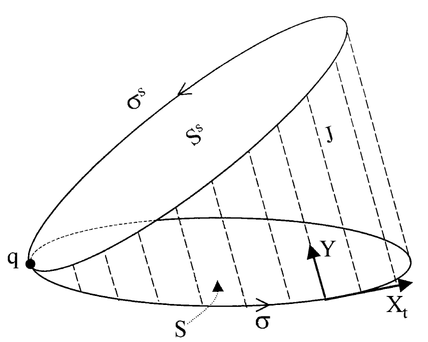

Fig. 1. Surface determined by the curves σ^1.

Moreover

$$\int_J \omega = s \int_0^1 \omega(X_t(\sigma(t)),\ Y_t(\sigma(t)))\,\mathrm{d}t + \mathrm{O}(s^2). \tag{3.11}$$

On the other hand, for a given $t \in [0, 1]$

$$\left(\frac{\mathrm{d}}{\mathrm{d}s} f_t^s(\sigma^s(t))\right)\bigg|_{s=0} = \left(\frac{\partial}{\partial s} f_t^s(\sigma(t))\right)\bigg|_{s=0} + Y_t(\sigma(t))(f_t). \tag{3.12}$$

We set

$$\dot{f}_t(p) := \left(\frac{\partial}{\partial s} f_t^s(p)\right)\bigg|_{s=0}.$$

As $\iota_{X_t}\omega = -\mathrm{d} f_t$, from (3.12) it follows:

$$\frac{\mathrm{d}}{\mathrm{d}s}\bigg|_{s=0} \int_0^1 f_t^s(\sigma^s(t))\,\mathrm{d}t = \int_0^1 \dot{f}_t(\sigma(t))\,\mathrm{d}t - \int_0^1 \omega(X_t(\sigma(t)), Y_t(\sigma(t)))\,\mathrm{d}t. \tag{3.13}$$

By (3.10),(3.11) and (3.13)

$$\Delta(s) - \Delta(0) = -s \int_0^1 \dot{f}_t(\sigma(t))\,\mathrm{d}t + \mathrm{O}(s^2) \quad (\text{modulo}\,\mathbb{Z}),$$

so

$$\kappa(\psi^s) - \kappa(\psi) = -2\pi \mathrm{i} s \kappa(\psi) \int_0^1 \dot{f}_t(\sigma(t))\,\mathrm{d}t + \mathrm{O}(s^2),$$

and finally

$$\left(\frac{\mathrm{d}}{\mathrm{d}s}\kappa(\psi^s)\right)\bigg|_{s=0} = -2\pi \mathrm{i}\kappa(\psi) \int_0^1 \dot{f}_t(\psi_t(q))\,\mathrm{d}t. \tag{3.14}$$

By $\mathcal{L}$ is denoted the space of C^1-loops in $\mathrm{Ham}(M)$ based at id, i.e., $\mathcal{L}$ is the space of isotopies ending at id. Given $\psi \in \mathcal{L}$, let ψ^s a curve in $\mathcal{L}$ with $\psi^0 = \psi$. For each s one has the corresponding time-dependent Hamiltonian function f_t^s. The tangent vector Z defined by ψ^s is determined by the family of functions

$$\dot{f}_t := \frac{\partial}{\partial s}\bigg|_{s=0} f_t^s,$$

which in turn can be identified with the corresponding Hamiltoninan family of vector fields.

On $\mathcal{L}$ we define the 1-form Ω as follows. Given $Z \in T_\psi \mathcal{L}$, determined by the family $\{\dot{f}_t\}$,

$$\Omega_\psi(Z) := \int_0^1 \dot{f}_t(\psi_t(q))\,\mathrm{d}t, \tag{3.15}$$

where q is any point of M. The left-hand side in (3.14) is independent of the point q, and so the right-hand side is also; therefore Ω is well-defined.

If $\Omega = 0$, then for any loop ψ in $\mathrm{Ham}(M)$ and any deformation ψ^s of ψ we have

$$\left(\frac{\mathrm{d}}{\mathrm{d}s}\kappa(\psi^s)\right)\Big|_{s=0} = 0,$$

and conversely. In this case κ is invariant under homotopies.

The Lie algebra of the group $\mathrm{Ham}(M)$ consists of all smooth functions on M which satisfy the normalization condition. The prequantization map $\mathcal{P}$ is a representation of this algebra, as we said in Section 2. In general $\mathcal{P}$ is not the tangent representation of one representation of $\widetilde{\mathrm{Ham}}(M)$, the universal cover of $\mathrm{Ham}(M)$. In the following we analyze this issue. An element of $\widetilde{\mathrm{Ham}}(M)$ is a homotopy class of a curve in $\mathrm{Ham}(M)$ which starts at id, i.e., the homotopy class $[\psi]$ of a Hamiltonian isotopy ψ. When Ω vanishes $\mathcal{T}_\psi$ depends only on the homotopy class $[\psi]$, this fact allows to construct a representation of $\widetilde{\mathrm{Ham}}(M)$ whose tangent representation is $\mathcal{P}$.

Proposition 14. *If $\Omega = 0$, then the prequantization map $\mathcal{P}$ extends to a representation of $\widetilde{\mathrm{Ham}}(M)$.*

Proof. Given the isotopy ψ, let $\{\psi^s\}$ be a deformation of ψ. For each s the path ζ^s in $\mathrm{Ham}(M)$ defined as the usual product path $\psi^s \cdot \psi^{-1}$ of the corresponding paths is a closed isotopy. Since $\Omega = 0$,

$$\left(\frac{\mathrm{d}}{\mathrm{d}s}\kappa(\zeta^s)\right)\Big|_{s=0} = 0.$$

As

$$(\mathcal{T}_{\psi^{-1}} \circ \mathcal{T}_{\psi^s})(\rho) = \mathcal{T}_{\psi^s\cdot\psi^{-1}}(\rho) = \kappa(\psi^s \cdot \psi^{-1})\rho$$

for every $\rho \in \Gamma(L)$, then

$$\frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0}(\mathcal{T}_{\psi^{-1}}\mathcal{T}_{\psi^s})(\rho) = \left(\frac{\mathrm{d}}{\mathrm{d}s}\kappa(\zeta^s)\right)\Big|_{s=0}\rho = 0.$$

So the transport $\mathcal{T}_\psi$ along ψ depends only on the homotopy class $[\psi]$, i.e., $\mathcal{T}$ is well-defined on $\widetilde{\mathrm{Ham}}(M)$.

If ψ and χ are isotopies, one can consider $\chi\circ\psi$, the isotopy defined by $(\chi\circ\psi)_t = \chi_t\circ\psi_t$. On the other hand, one has the juxtaposition $\psi\star\chi$ given by $(\psi\star\chi)_t = \psi_{2t}$ for $t \in [0, 0.5]$ and $(\psi\star\chi)_t = \chi_{2t-1}\circ\psi_1$ for $t \in [0.5, 1]$. As $[\chi\circ\psi] = [\psi\star\chi]$ (see [8]), we have

$$\mathcal{T}_{[\chi][\psi]} = \mathcal{T}_{[\psi\star\chi]} = \mathcal{T}_{[\chi]}\circ\mathcal{T}_{[\psi]}.$$

Hence $\mathcal{T}$ is a representation of $\widetilde{\mathrm{Ham}}(M)$ and, by construction, its tangent representation is $\mathcal{P}$. □

In a similar way one can prove the following theorem.

Theorem 15. *Let (M, ω) be a compact, quantizable manifold. The following properties are equivalent:*

1. *The* 1*-form* Ω *vanishes.*
2. *For any simply connected Lagrangian submanifold N of M and every loop* ψ_t *in* $\mathrm{Ham}(M)$ *at* id *the Berry phase of the loop* $\{\psi_t(N)\}_t$ *of Lagrangian submanifolds depends only on the homotopy class of* ψ_t.
3. *Given an arbitrary foliation F of M and an arbitrary Hamiltonian isotopy* ψ, *the natural identifications of* $\mathcal{Q}_F$ *and* $\mathcal{Q}_{\psi_1(F)}$ *defined by* $\mathcal{T}_\psi$ *and* $\mathcal{T}_{\psi'}$ *are equal for all* $\psi' \in [\psi]$.

Proof. We assume (1). If ψ^s is a deformation of $\psi \in \mathcal{L}$, as in the foregoing proposition $\psi^s \cdot \psi^{-1}$ is a closed isotopy. By (1)

$$\left(\frac{\mathrm{d}\kappa(\psi^s \cdot \psi^{-1})}{\mathrm{d}s}\right)\bigg|_{s=0} = 0.$$

From (3.8) and Theorem 5 it follows property (2).

Conversely, let ψ be an element of $\mathcal{L}$ and ψ^s an arbitrary curve in $\mathcal{L}$ with $\psi^0 = \psi$. This curve defines a deformation of ψ. Let us take $\tau \in \mathcal{Q}_F$ with $\tau_{|N} \neq 0$, for N a leaf of a Lagrangian foliation F. By (2) and Theorem 5 $\kappa(\psi)\tau_{|N} = \kappa(\psi^s)\tau_{|N}$ for all s. Therefore $(\mathrm{d}\kappa(\psi^s)/\mathrm{d}s)|_{s=0} = 0$, consequently $\Omega_\psi = 0$. □

Next we study a particular case: the behavior of $\kappa(\psi)$ under deformations consisting of 1-parameter subgroups. Let us suppose that ψ^s for each s is a 1-periodic Hamiltonian flow, then f_t^s is independent of t and we put $f_t^s = f^s$. One defines the function $\dot{f}$ by $\dot{f}(p) = ((\mathrm{d}/\mathrm{d}s)f^s(p))|_{s=0}$. As $\{\sigma^s(t)|t \in [0, 1]\}$ is an integral curve for the Hamiltonian function f^s

$$f^s(\sigma^s(t)) = f^s(q) = (f + s\dot{f})(q) + \mathrm{O}(s^2) = f(q) + s\dot{f}(q) + \mathrm{O}(s^2).$$

On the other hand

$$f^s(\sigma^s(t)) = (f + s\dot{f})(\sigma^s(t)) + \mathrm{O}(s^2) = f(q) + s(Y_t(\sigma(t))(f) + \dot{f}(\sigma(t))) + \mathrm{O}(s^2).$$

Therefore

$$\dot{f}(q) = Y_t(\sigma(t))(f) + \dot{f}(\sigma(t)).$$

Now $\mathrm{d}f = -\iota_X\omega$, then

$$\int_0^1 \dot{f}(\sigma(t))\,\mathrm{d}t = \dot{f}(q) + \int_0^1 \omega(X(\sigma(t)), Y_t(\sigma(t)))\,\mathrm{d}t.$$

The symplectomorphism $\delta := \psi_t^s \circ \psi_t^{-1}$ applies the curve $\sigma(t)$ into $\sigma^s(t)$. Hence $\delta(S)$ is a surface whose boundary is $\sigma^s(t)$ and

$$\int_{S^s} \omega = \int_S \delta^*\omega = \int_S \omega. \tag{3.16}$$

From (3.16), (3.10) and (3.11) it follows

$$\int_0^1 \omega(X(\sigma(t)), Y_t(\sigma(t)))\,\mathrm{d}t = 0.$$

From (3.14) it follows:

$$-\frac{1}{2\pi \mathrm{i}\kappa(\psi)}\left(\frac{\mathrm{d}}{\mathrm{d}s}\kappa(\psi^s)\right)\bigg|_{s=0} = \dot{f}(q). \tag{3.17}$$

As the left-hand side in (3.17) is independent of the point q, it turns out that $\dot{f}$ is constant on M. The normalization condition of each f^s implies

$$0 = \int_M f^s \omega^n = \int_M (f + s\dot{f})\omega^n + \mathrm{O}(s^2) = s\dot{f}\int_M \omega^n + \mathrm{O}(s^2).$$

Hence $\dot{f} \equiv 0$, and by (3.14)

$$\left(\frac{\mathrm{d}}{\mathrm{d}s}\kappa(\psi^s)\right)\bigg|_{s=0} = 0.$$

One has

Theorem 16. *κ is invariant under homotopies consisting of 1-parameter subgroups in M.*

Corollary 17. *Let ψ and ψ' be 1-periodic Hamiltonian flows generated by the Hamiltonian functions f and f', respectively. If ψ and ψ' are homotopic in the space of 1-parameter subgroups, then*

$$f(p) = f'(p') \pmod{\mathbb{Z}}$$

for p and p' critical points of f and f', respectively.

Proof. It is a consequence of Theorem 16 and Corollary 12. □

4. A grading in $\pi_2(\mathrm{Ham}(M))$

We will prove in this Section that the 1-form Ω on $\mathcal{L}$ is closed. If $\phi := \{\phi^s\}$ is a closed curve in $\mathcal{L}$, one can consider the map $\kappa(\phi^-) : s \in S^1 \mapsto \kappa(\phi^s) \in U(1)$, its winding number is

$$\deg(\kappa(\phi^-)) = \int_{S^1} \frac{1}{2\pi \mathrm{i}\kappa(\phi^s)} \frac{\mathrm{d}\kappa(\phi^s)}{\mathrm{d}s}\, \mathrm{d}s.$$

By (3.14) and (3.15) this winding number is equal to

$$-\int_{S^1} \Omega_{\phi^s}(\dot{\phi}^s) ds,$$

where $\dot{\phi}^s$ is the vector of $T_{\phi^s}\mathcal{L}$ defined by the curve $\{\phi^s\}_s$.

If ϕ and ξ are two homotopic loops in $\mathcal{L}$, then there is a homotopy ${}_r\phi^s$ such that ${}_0\phi^s = \phi^s$ and ${}_1\phi^s = \xi^s$. Therefore $\kappa({}_r\phi^-)$ is a homotopy between the maps $\kappa(\phi^-)$ and $\kappa(\xi^-)$, so these maps have the same degree (see [4, p. 129]).

If ϕ and φ are loops in $\mathcal{L}$ based at the same point and $\zeta = \phi \cdot \varphi$ is the path product, then $\deg(\kappa(\zeta^-))$ is equal to

$$\frac{1}{2\pi \mathrm{i}} \left(\int_0^{0.5} \frac{1}{\kappa(\phi^{2s})} \frac{\mathrm{d}\kappa(\phi^{2s})}{\mathrm{d}s}\, \mathrm{d}s + \int_{0.5}^{1} \frac{1}{\kappa(\varphi^{2s-1})} \frac{\mathrm{d}\kappa(\varphi^{2s-1})}{\mathrm{d}s}\, \mathrm{d}s \right),$$

and this expression is equal to $\deg(\kappa(\phi^-)) + \deg(\kappa(\varphi^-))$. Thus, we have the following theorem.

Theorem 18. *Ω defines an element of $H^1(\mathcal{L}, \mathbb{Z})$. Moreover, if ϕ is a closed curve on $\mathcal{L}$ then $-\Omega([\phi])$ is the degree of the map $\kappa(\phi^-)$.*

We denote by c the loop in $\mathrm{Ham}(M)$ defined by $c(s) = \mathrm{id}$ for all s. Since $\pi_1(\mathcal{L}, c) = \pi_2(\mathrm{Ham}(M), \mathrm{id})$, the form Ω defines a *degree* on $\pi_2(\mathrm{Ham}(M), \mathrm{id})$. Given $[\phi] \in \pi_2(\mathrm{Ham}(M), \mathrm{id})$

$$\mathrm{Deg}([\phi]) := \Omega([\phi]) = -\deg(\kappa(\phi^-)). \tag{4.1}$$

As Deg is a homomorphism, this grading on $\pi_2(\mathrm{Ham}(M))$ is compatible with the group structure.

If Ω is exact, then $\mathrm{Deg} = 0$. In this case there is a potential map $H : \mathcal{L} \to \mathbb{R}$ such that, if $\{\nu^s\}_s$ is a curve in $\mathcal{L}$ starting at $c \in \mathcal{L}$

$$H(\nu^s) = \int_0^s \Omega_{\nu^a}(\dot{\nu}^a)\, \mathrm{d}a = -\frac{1}{2\pi \mathrm{i}} \int_0^s \frac{1}{\kappa(\nu^a)} \frac{\mathrm{d}\kappa(\nu^a)}{\mathrm{d}a}\, \mathrm{d}a.$$

So

$$\frac{\mathrm{d}H(\nu^s)}{\mathrm{d}s} = -\frac{1}{2\pi \mathrm{i}\kappa(\nu^s)} \frac{\mathrm{d}\kappa(\nu^s)}{\mathrm{d}s}.$$

By (3.6) $\kappa(c) = 1$, so $\kappa(\nu^s) = \exp(-2\pi \mathrm{i} H(\nu^s))$. Hence for every $\psi \in \mathcal{L}$ that can be joined with c by a path, we have $\kappa(\psi) = \exp(-2\pi \mathrm{i} H(\psi))$. A similar expression holds in each connected component of $\mathcal{L}$. Thus H is a lifting of the action integral function $\mathcal{A} : \mathcal{L} \to \mathbb{R}/\mathbb{Z}$ to an $\mathbb{R}$-valued function.

Conversely, if there is a lifting of $\mathcal{A}$ to an $\mathbb{R}$-valued function, then $\mathrm{Deg} = 0$, i.e., Ω is exact. In short, it is dealt in the following proposition.

Proposition 19. *The class $[\Omega] \in H^1(\mathcal{L}, \mathbb{Z})$ is the obstruction to existence of a lifting of $\mathcal{A}$ to an $\mathbb{R}$-valued function.*

A generic element of $\pi_2(\mathrm{Ham}(M), \mathrm{id})$ is given by a map $\phi = (\phi_t^s)$ from I^2 into $\mathrm{Ham}(M)$, such that for each $s\phi^s = \{\phi_t^s\}_t$ is a Hamiltonian isotopy ending at id, defined by the normalized time-dependent Hamiltonian f_t^s. One can also consider a family of particular elements in $\pi_2(\mathrm{Ham}(M), \mathrm{id})$, those χ such that for each s, χ^s is the Hamiltonian flow associated to a Hamiltonian function. One has the following result.

Proposition 20. *If* $[\phi] = [\chi] \in \pi_2(\mathrm{Ham}(M), \mathrm{id})$, *then*

$$\int_0^1 \int_0^1 \left(\frac{\partial f_t^s}{\partial s}\right)(\phi_t^s(q))\,\mathrm{d}t\,\mathrm{d}s = 0$$

for every $q \in M$.

Proof.

$$\Omega([\chi]) = \Omega([\phi]) = \int_0^1 \Omega_{\phi^s}(\dot{\phi}^s)\,\mathrm{d}s. \tag{4.2}$$

By (3.15)

$$\Omega_{\phi^s}(\dot{\phi}^s) = \int_0^1 \left(\frac{\partial f_t^s}{\partial s}\right)(\phi_t^s(q))\,\mathrm{d}t \tag{4.3}$$

for every $q \in M$.

On the other hand, $\kappa(\chi^s)$ is independent of s by Theorem 16. So the map $\kappa(\chi^-)$ has degree 0. The proposition follows from Theorem 18, (4.2) and (4.3). □

5. Example: coadjoint orbits of *SU(2)*

We will check the above results when M is a coadjoint orbit [6] of the group $SU(2)$. Let η be the element of $\mathfrak{su}(2)^*$

$$\eta : \begin{pmatrix} a\mathrm{i} & w \\ -\bar{w} & -a\mathrm{i} \end{pmatrix} \in \mathfrak{su}(2) \to ka \in \mathbb{R},$$

where k is a non-zero real number. The subgroup of isotropy G_η of η is the subgroup $U(1)$ of $SU(2)$. So the coadjoint orbit $\mathcal{O}_\eta$ of η can be identified with $SU(2)/U(1) = S^2$. If $\mu \in \mathcal{O}_\eta$ then $\mu = g \cdot \eta$ with

$$g = \begin{pmatrix} x & y \\ -\bar{y} & \bar{x} \end{pmatrix} \in SU(2). \tag{5.1}$$

If we put

$$x = \cos(\tfrac{1}{2}\theta)\exp(\mathrm{i}\phi_1), \qquad y = \sin(\tfrac{1}{2}\theta)\exp(-i\phi_2) \quad \text{with } 0 \le \theta \le \pi, \tag{5.2}$$

then the point in S^2 corresponding to $\mu \in \mathcal{O}_\eta$ through the diffeomorphism $\mathcal{O}_\eta \simeq SU(2)/U(1) \simeq S^2$ has the spherical coordinates $(\theta, \phi = \phi_1 - \phi_2)$.

On the other hand, $\mathfrak{su}(2) = \mathbb{R}A \oplus \mathbb{R}B \oplus \mathbb{R}Z$ with

$$A = \begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad Z = \begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix}.$$

The invariant vector fields X_A, X_B generated by $A, B \in \mathfrak{su}(2)$ can be expressed in terms of the fields $\partial/\partial\theta$, $\partial/\partial\phi$. Given $\mu \in \mathcal{O}_\eta$, $X_B(\mu)$ is defined by the curve $\mathrm{e}^{tB}\mu$. If $\mu = g\eta$, with g as above, then $\mathrm{e}^{tB}g$ is the element of $SU(2)$ determined by the pair

$$(x', y') = (x\cos t - \bar{y}\sin t,\ y\cos t + \bar{x}\sin t).$$

An easy but tedious calculation shows that

$$(x', y') = (\cos(\tfrac{1}{2}\theta')\,\mathrm{e}^{\mathrm{i}\phi_1'},\ \sin(\tfrac{1}{2}\theta')\,\mathrm{e}^{-i\phi_2'}) + \mathrm{O}(t^2)$$

with $\theta' = \theta + 2t\cos\phi,\ \phi_1' = \phi_1 + t\tan(\frac{1}{2}\theta)\sin\phi,\ \phi_2' = \phi_2 + t\cot(\frac{1}{2}\theta)\sin\phi$. Therefore

$$X_B(\theta, \phi) = 2\cos\phi\frac{\partial}{\partial\theta} - 2\cot\theta\sin\phi\frac{\partial}{\partial\phi}. \tag{5.3}$$

Similarly,

$$X_A(\theta, \phi) = 2\sin\phi\frac{\partial}{\partial\theta} + 2\cot\theta\cos\phi\frac{\partial}{\partial\phi}. \tag{5.4}$$

The symplectic structure on $\mathcal{O}_\eta$ is defined by the form ω, whose action on invariant vector fields is

$$\omega_\mu(X_C(\mu), X_D(\mu)) = \mu([C, D]).$$

ω can also be expressed in the spherical coordinates. With the above notations

$$\omega_\mu(X_A, X_B) = \eta(g^{-1}[A, B]g) = -2k(|x|^2 - |y|^2) = -2k\cos\theta.$$

Using (5.3) and (5.4) a simple calculation gives

$$\omega = \tfrac{1}{2}k\sin\theta\,\mathrm{d}\theta \wedge \mathrm{d}\phi. \tag{5.5}$$

Given $C \in \mathfrak{su}(2)$, the function h_C on $\mathcal{O}_\eta$ defined by $h_C(\mu) = \mu(C)$ satisfies $\omega(X_C, .) = \mathrm{d}h_C$. In spherical coordinates

$$h_A(\theta, \phi) = -k\sin\theta\cos\phi, \qquad h_B(\theta, \phi) = k\sin\theta\sin\phi. \tag{5.6}$$

Henceforth, we assume that $k = n/2\pi$ with $n \in \mathbb{Z}$. Then, the orbit $\mathcal{O}_\eta$ possesses an invariant prequantization (see [7]).

We can consider the family $\{\psi_t\}$ of symplectomorphisms of $\mathcal{O}_\eta$ defined by $\psi_t(\mu) := \mathrm{e}^{tA} \cdot \mu$. As

$$\mathrm{e}^{tA} = \begin{pmatrix} \cos t & \mathrm{i}\sin t \\ \mathrm{i}\sin t & \cos t \end{pmatrix}, \tag{5.7}$$

hence $\psi_\pi : S^2 \to S^2$ is the identity, and $\psi = \{\psi_t | t \in [0, \pi]\}$ is a loop in the group of $\mathrm{Ham}(\mathcal{O}_\eta)$.

If one takes the north pole $p(\theta = 0, \phi = 0)$, the curve $\psi_t(p)$ is the path obtained as product of the paths defined by the meridians $\phi = \pi/2$ and $\phi = 3\pi/2$. So by (5.6) $h_A(\psi_t(p)) = 0$, and

$$S = \{(\theta, \phi) | \pi/2 \le \phi \le 3\pi/2,\ \ \theta \in [0, \pi]\}$$

oriented with $\mathrm{d}\theta \wedge \mathrm{d}\phi$ is an oriented surface whose boundary is the curve $\psi_t(p)$. By (5.5) $\int_S \omega = k\pi$, and from (3.6) we obtain $\kappa_p(\psi) = (-1)^n$.

We could calculate $\kappa_q(\psi)$ for $q(\theta = \pi/2, \phi = 0)$. Now $\psi_t(q) = q$ for all t, hence the integral of ω in (3.6) vanishes. $h_A(q) = -n/2\pi$, consequently $-\int_0^\pi f_t(\psi_t(q)) = -\frac{1}{2}n$, and $\kappa_q(\psi) = (-1)^n$.

Let us consider the point $r = (\pi/2, \pi/2) \in S^2$, according to (5.2) this point can be represented by the element of $g \in SU(2)$ defined by $x = 2^{-1/2}\mathrm{i}$, $y = 2^{-1/2}$. Denoting by (θ', ϕ') the spherical coordinates of $\psi_t(r)$, from (5.7) one deduces

$$\mathrm{e}^{\mathrm{i}\phi'} \cos(\tfrac{1}{2}\theta') = \frac{\mathrm{i}}{\sqrt{2}}(\cos t - \sin t), \qquad \sin(\tfrac{1}{2}\theta') = \frac{1}{\sqrt{2}}(\cos t + \sin t).$$

Hence $\theta' = 2t + \pi/2$, $\phi' = \pi/2$ when $t \in [0, \pi/4]$, etc., i.e., $\{\psi_t(r)\}$ is the union of the meridians $\phi = \pi/2$ and $\phi = 3\pi/2$. So $h_A(\psi_t(r)) = 0$. On the other hand

$$\int_0^\pi \int_{\pi/2}^{3\pi/2} \frac{n}{4\pi} \sin\theta \, \mathrm{d}\theta \wedge \mathrm{d}\phi = \frac{n}{2}.$$

So $\kappa_r(\psi) = (-1)^n$.

The equalities $\kappa_p(\psi) = \kappa_q(\psi) = \kappa_r(\psi)$ can also be considered as a checking of Theorem 10.

We will determine $\kappa(\chi)$, when χ_t is the symplectomorphism of S^2 given by $\chi_t(q) = \mathrm{e}^{t(aA+bB)}q$, where $a, b \in \mathbb{R}$. For $t \geq 0$ we put $c = t(b + a\mathrm{i})$, so

$$t(aA + bB) = \begin{pmatrix} 0 & c \\ -\bar{c} & 0 \end{pmatrix}.$$

If we define $\epsilon := c/|c|$, it is easy to deduce

$$\mathrm{e}^{t(aA+bB)} = \begin{pmatrix} \cos|c| & \epsilon \sin|c| \\ -\bar{\epsilon}\sin|c| & \cos|c| \end{pmatrix} \tag{5.8}$$

For $t_1 = \pi/\sqrt{a^2 + b^2}$ the Hamiltonian symplectomorphism $\chi_{t_1} = \mathrm{id}$, so $\{\chi_t | t \in [0, t_1]\}$ is a loop in $\mathrm{Ham}(S^2)$. From now on we assume $\sqrt{a^2 + b^2} = 1$, then $\chi_\pi = \mathrm{id}$.

Let p be the north pole, then $\chi_t(p)$ is the point which corresponds to the pair

$$(x = \cos t, \ y = \epsilon \sin t) \tag{5.9}$$

in the notation (5.1). We put $\epsilon = \mathrm{e}^{\mathrm{i}\alpha}$, from (5.2) and (5.9) it follows that the spherical coordinates of $\chi_t(p)$ are $(2t, \alpha)$ for $t \in [0, \pi/2]$.

Similarly, when t runs on $[\pi/2, \pi]$ the point $\chi_t(p)$ runs on the meridian $\phi = \pi + \alpha$ from $\theta = \pi$ to $\theta = 0$, i.e., $\chi_t(p) = (2\pi - 2t, \pi + \alpha)$.

As $h_{aA+bB} = ah_A + bh_B$ and $\epsilon = \cos\alpha + \mathrm{i}\sin\alpha$, by (5.6)

$$h_{aA+bB}(\theta, \phi) = k \sin\theta \sin(\phi - \alpha).$$

Taking into account the spherical coordinates of $\chi_t(p)$ determined above, one deduces $h_{aA+bB}(\chi_t(p)) = 0$ for every $t \in [0, \pi]$. Thus $\kappa(\chi) = \exp(2\pi\mathrm{i}\int_S \omega)$, where S is the hemisphere limited by the meridian $\phi = \alpha$ and $\phi = \pi + \alpha$. Therefore $\kappa(\chi) = (-1)^n$. In summary it can be given as the following theorem.

Theorem 21. *Let η be the element of $\mathfrak{su}(2)^*$ defined by $\eta\begin{pmatrix} a\mathrm{i} & w \\ -\bar{w} & -a\mathrm{i} \end{pmatrix} = (n/2\pi)a$ with $n \in \mathbb{Z}$. If χ is a loop in $\mathrm{Ham}(\mathcal{O}_\eta)$ which is a 1-parameter subgroup generated by an invariant vector field, then $\kappa(\chi) = (-1)^n$.*

The vector $aA + bB \in \mathfrak{su}(2)$ with $a^2 + b^2 = 1$ can be deformed by means of a rotation into $a'A + b'B$, if $(a')^2 + (b')^2 = 1$. If we denote $\chi'_t := \exp(t(a'A + b'B))$, by Theorem 16 $\kappa(\chi) = \kappa(\chi')$. Therefore, Theorem 21 can also be considered as a checking of Theorem 16.

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